

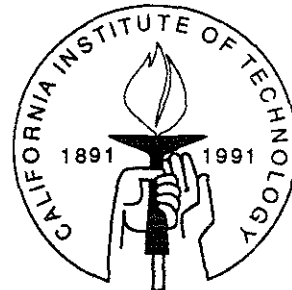
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Arbitrage Restrictions Across Financial Markets: Theory,
Methodology and Tests

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Abstract

The Cox, Ingersoll and Ross [1985a] general equilibrium model is extended by allowing the representative investor to trade in a batch call option market with execution price uncertainty. Necessary restrictions on the execution price uncertainty for the original equilibrium to remain intact are derived. They take the form of moment conditions in the pricing error (defined as the difference between the observed call price and the theoretical call price that would obtain in the absence of execution price uncertainty). The moment conditions can easily be estimated and tested using a version of the Method of Simulation Moments (MSM). In it, simulation estimates, obtained by discretely approximating the risk-neutral processes of the underlying stock price and the interest rate, are substituted for analytically unknown call prices. The asymptotics and other aspects of the MSM estimator are discussed. The model is tested on transaction prices from the Berkeley Options Data Base.

Arbitrage Restrictions Across Financial Markets: Theory, Methodology and Tests

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1 Introduction¹

Since the seminal Black and Scholes [1973] paper there have been numerous empirical studies of call option pricing.² Various sorts of mispricing of the original Black-Scholes formula have been discovered, and extensions have not always proved to be fruitful.

In this paper we shall report results from testing a call pricing formula that is consistent with the Cox, Ingersoll and Ross [1985a], [1985b] general equilibrium framework. While it has not yet been tested, this model is promising because, as Bailey and Stulz [1989] have shown, it might explain some of the biases previously encountered with the Black-Scholes model.

The Black-Scholes model was derived under the assumption of homoscedasticity and constant interest rates. The Cox-Ingersoll-Ross model features heteroscedasticity and stochastic interest rates. The latter leads to different call prices when the correlation between the changes in the price of the underlying asset and changes in the interest rates is *nonzero*.

If this correlation is zero, an increase in interest rates has two effects that more or less cancel: (i) a value-decreasing effect caused by an increase in the discount rate, (ii) a value-enhancing effect caused by an increase in the volatility of the price of the underlying asset.³ If the correlation is positive, increases in the interest rates will often

¹This paper combines and extends results reported in earlier working papers of ours. Comments from several colleagues are gratefully acknowledged. We remain responsible, however, for any remaining mistakes. We thank Carnegie Mellon University and INSEAD for financial support, and the JPL/Caltech Supercomputing Project for computing support.

²See Galai [1983], or Hull [1989], section 12.11, for a review.

³In the Cox-Ingersoll-Ross model, the volatility of primary assets is proportional to the square root of the interest rate. See Section 2 for details.

be accompanied with increases in the price of the underlying asset. Consequently, a call option will be worth more than under zero correlation. The opposite occurs when the correlation is negative.

Unfortunately, the Black-Scholes or Cox-Ingersoll-Ross call pricing formulae, or any other traditional option pricing model for that matter, are easily rejected. It suffices to back parameter values out of a limited set of observed option prices, and to find an additional option price which does not exactly match the corresponding theoretical price at these parameter values. This deficiency is caused by a crucial assumption of traditional models, namely that the option price path can be duplicated perfectly. This allowed pricing *by arbitrage*, an analytically attractive concept, but at the same time generated empirically untenable results. While the shortcomings of traditional modeling have often been acknowledged in the empirical literature, they have, with the exception of Lo [1986], not been addressed properly.

Lo [1986] relates pricing errors (the discrepancies between observed prices and theoretical prices) to errors in the estimation of the parameters of the model. We attempt an alternative route. In particular, we change the call option market in the economy of Cox, Ingersoll and Ross [1985a] to a batch market with execution price uncertainty (as in Ho, Schwartz and Whitcomb [1985]), in which the clearing price will not be known at the time orders are submitted. The execution price uncertainty is assumed to be *exogenous*. ~~In particular, it is not insurable through perhaps complicated dynamic trading strategies.~~ Nevertheless, its nature will be left unspecified, in order not to lose robustness. We merely want to investigate the restrictions that it must satisfy for the original Cox-Ingersoll-Ross equilibrium to be unaffected. In other words, the representative investor will be provided with an additional trading opportunity, and the question will be posed: when does he/she not wish to use it (thereby leaving his/her consumption choice unaltered)?

We shall focus on a simple zero-investment trading strategy involving the batch call option market and the market of the underlying security. This strategy will be risky, since the execution price uncertainty is uninsurable. Hence, we shall call it *risk arbitrage*. We then employ the standard result that, for the investor to be indifferent to a risky zero-investment trading opportunity, its payoff has to be orthogonal to the marginal rate of substitution of consumption. We evaluate the latter at the consumption path chosen in the original Cox-Ingersoll-Ross economy.

Consequently, we shall investigate restrictions on the prices in the call option market relative to the prices in the market for the underlying asset that make it worthless for the representative investor to engage in risk arbitrage across the two markets, assuming that he/she chooses his/her consumption as in the original Cox-Ingersoll-Ross economy.

The pricing restrictions we shall obtain are moment restrictions on the pricing errors. The latter are defined as the option clearing price minus the theoretical option price evaluated at the synchronous price of the underlying asset. The pricing formula used to

calculate the theoretical option price is identical to the one that obtains in the absence of execution price uncertainty.

The moment restrictions would have been readily testable using Generalized Method of Moments (GMM) if an analytical expression for the theoretical call price was known. As in Boyle [1977], we shall use Monte Carlo simulation (with variance reduction) in order to estimate theoretical call prices. Moreover, in the absence of an analytical expression for the conditional risk-neutral joint distribution of the future stock price and interest rate, we shall obtain simulates by discretizing the corresponding continuous-time processes. Talay and Tubaro's [1989] Romberg interpolation will be employed to minimize any biases.

The substitution of simulation estimates for analytical values in the estimation of moment conditions has become known as Method of Simulated Moments (MSM) (see McFadden [1989] and Pakes and Pollard [1989]). Whereas Hansen's [1982] arguments cannot readily be applied here to show consistency and asymptotic normality of the MSM estimator (because of a nondifferentiability in the payoff of a call option as a function of the parameters), McFadden's [1989] or Pakes and Pollard's [1989] arguments impose too restrictive i.i.d. assumptions to apply to the present case. Moreover, the biases caused by the approximations have to be taken into account. The paper explains how the asymptotics can be restored.

Results will be reported from bringing the model to the data in MSM tests on call option transaction prices taken from the Berkeley Options Data Base. Synchronous continuous-time interest rates were extracted from 90-day Treasury bill futures quotes (from the Chicago Mercantile Exchange), using a pricing formula consistent with the present economy (as provided in Cox, Ingersoll and Ross [1981]).

While the theory, methodology and tests of this paper focus on the pricing of call options written on common stock, it will be obvious how to extend the analysis to other "derivative assets" such as put options, futures and forward contracts, etc. It is not clear, however, how to extend the analysis to claims which, if they are exercised, will be exercised before maturity. Further research will have to be done to cover that case.

The remainder of the paper is organized as follows: Section 2 presents the theoretical model. Section 3 introduces the methodology (MSM) and discusses the asymptotics (details of which are in the Appendix). Section 4 presents the data, reports and interprets the results. Section 5 concludes.

2 The Theoretical Model

As already pointed out in the Introduction, we shall develop a model of arbitrage restrictions across financial markets by extending the Cox, Ingersoll and Ross [1985a][1985b]

framework allowing for execution price uncertainty. We consider an economy with a certain number of investment opportunities (indexed by i), the instantaneous returns of which follow Itô processes:

$$\frac{ds_i(t)}{s_i(t)} = \mu_i x(t) dt + \sigma_i \sqrt{x(t)} dz_i(t), \quad (1)$$

where $dz_i(t)$ denotes the instantaneous increment of a standard Brownian motion, and $x(t)$ is a mean-reverting state variable:

$$dx(t) = \alpha(\beta - x(t))dt + \sigma_x \sqrt{x(t)} dz_x(t) \quad (2)$$

Let ρ_i denote the instantaneous correlation between the standard Brownian motions in (1) and (2), i.e., $\rho_i = E[dz_i(t)dz_x(t)]$.

We assume the existence of a representative investor with logarithmic preferences. Given the investment opportunities of equation (1), the representative investor's wealth $I(t)$ (the market index) will change over time as follows:

$$\frac{dI(t)}{I(t)} = \mu_I x(t) dt + \sigma_I \sqrt{x(t)} dz_I(t) \quad (3)$$

Assuming that a risk-free asset is in zero net supply, the instantaneous risk-free rate, $r(t)$, will be proportional to the state variable, $x(t)$:

$$r(t) = \mu_I x(t) - \sigma_I^2 x(t) \quad (4)$$

Hence,

$$dr(t) = \alpha(\beta' - r(t))dt + \sigma_x \sqrt{r(t)} dz_x(t), \quad (5)$$

with

$$\beta' = \frac{\beta}{\mu_I - \sigma_I^2}$$

As in Cox, Ingersoll and Ross [1985a], we add financial markets to our economy. The payoffs on securities in those markets are contingent on the payoffs in the primary

markets. Financial markets are redundant markets in the sense that the securities in those markets are in zero net supply.

In particular, consider a financial market in which call options are traded, written on primary asset i . The call options mature in τ periods. Their exercise price equals k . They are European, i.e., they can be exercised only at maturity. Their payoff will equal $\max(0, s_i(t + \tau) - k)$. For the representative investor not to hold a nonzero quantity of these call options, their price must be:

$$c(s_i(t), r(t), k, \tau) = k E_{r(t), s'_i(t)} [e^{-\int_t^{t+\tau} r(u) du} \max(s'_i(t + \tau) - 1, 0)], \quad (6)$$

where, for u in $[t, t + \tau]$:

$$\frac{ds'_i(u)}{s'_i(u)} = r(u) du + \sigma'_i \sqrt{r(u)} d\tilde{z}_i(u),$$

$$s'_i(t) = \frac{s_i(t)}{k},$$

$$\sigma'_i = \frac{\sigma_i}{\mu_I - \sigma_I^2},$$

$$dr(u) = \alpha(\beta' - r(u)) du + \sigma_x \sqrt{r(u)} dz_x(u).$$

The expectation in (6) is taken over the *risk-neutral* process of the price of the underlying asset (driven by $d\tilde{z}_i(\cdot)$), which in general will not equal the corresponding true process displayed in equation (1) (driven by $dz_i(\cdot)$). In other words, it is not necessarily true that $d\tilde{z}_i = dz_i(\cdot)$. In the risk-neutral world, however, $d\tilde{z}_i(\cdot)$ will be increments of a standard Brownian motion, even if they are not in the true world. In the absence of knowledge about μ_i , we shall not be able to link $d\tilde{z}_i(\cdot)$ to realizations of $dz_i(\cdot)$. More concretely, in the empirical section we must not relate simulations of $d\tilde{z}_i(\cdot)$ to observed changes in the price of the underlying asset. Notice also that the homogeneity of the stochastic process of the price of the underlying asset is exploited in (6): the exercise price, k , multiplies the expectations operator, and the expectation is taken over the (risk neutral) process of the price of the underlying asset normalized by the exercise prices, $s'_i(\cdot)$. This will prove useful in the discussion of the asymptotic properties of our estimation strategy.

We now change the structure of the call option market. Instead of a market that is synchronous in all respects to the primary markets, we now make it a batch market with execution price uncertainty, as in Ho, Schwartz and Whitcomb [1985]. In other words, the clearing price, which will be denoted $c_i^c(t, k, \tau)$, is unknown when orders have to be submitted. We assume that the execution price uncertainty is exogenous to the economy.

In particular, it cannot be insured. The implications of such a market structure will be investigated under the assumption that payments in the execution of orders are made in risk-free zero coupon bonds. Alternative means of payment could have been used, such as the underlying primary asset. This does not affect the conclusions.

We now ask what restrictions have to be imposed on the execution price uncertainty for the original Cox-Ingersoll-Ross equilibrium to remain intact. We answer this by investigating a simple zero-investment trading strategy which the representative investor could engage in and insisting that he/she is indifferent to doing so. The trading strategy will be risky, because the execution price uncertainty in the call option market cannot be insured. Hence, as mentioned in the Introduction, we shall call it *risk arbitrage*.

In particular, consider the following. Let $b(r(t), \tau)$ denote the time t price of a risk-free zero coupon bond that matures at $t + \tau$. At $t^- (< t)$, the representative investor can engage in the following risk arbitrage:

- (i) Take \$1.00 in risk-free zero coupon bonds that mature at $t + \tau$ (the maturity date of the option). Submit an order at t^- to buy one call option at t . The payoff of this portfolio at $t + \tau$ equals:

$$\max(0, s_i(t + \tau) - k) + \left(\frac{1}{b(r(t^-), \tau + (t - t^-))} - \frac{c_i^o(t, k, \tau)}{b(r(t), \tau)} \right).$$

- (ii) Take \$1.00 in risk-free zero coupon bonds. At time t , set aside a certain number of dollars with which to replicate the payoff on a call option, i.e., with which to generate the payoff $\max(0, s_i(t + \tau) - k)$ at time $t + \tau$, by continuously rebalancing a portfolio of the underlying asset and bonds. Traditional option pricing arguments provide an expression for the number of dollars that need to be set aside to implement this self-financing option-replicating strategy, namely $c(s_i(t), r(t), k, \tau)$, as given in equation (6). Assume the remainder is kept in risk-free zero coupon bonds. The payoff on this portfolio at $t + \tau$ equals:

$$\max(0, s_i(t + \tau) - k) + \left(\frac{1}{b(r(t^-), \tau + (t - t^-))} - \frac{c(s_i(t), r(t), k, \tau)}{b(r(t), \tau)} \right).$$

Construct a zero investment strategy by going long (ii) and shorting (i). The payoff on this strategy at $t + \tau$ equals:

$$(c_i^o(t, k, \tau) - c(s_i(t), r(t), k, \tau)) / b(r(t), \tau),$$

which is random from the vantage point of time t^- .

For the representative investor to be indifferent to this risk arbitrage its value has to equal zero. Using the standard result that investment opportunities will be valued in terms of marginal rates of substitution of consumption, it must be that:

$$0 = E_{t^-}[\lambda_{t^-}^{t+\tau}(c_i^o(t, k, \tau) - c(s_i(t), r(t), k, \tau))/b(r(t), \tau)],$$

where

$$\lambda_{t^-}^{t+\tau} = e^{-\int_{t^-}^{t+\tau} d \ln I(u)} = \frac{I(t^-)}{I(t+\tau)}$$

$\lambda_{t^-}^{t+\tau}$ is the marginal rate of substitution of consumption in t^- for consumption in $t+\tau$, which in the present case of a representative consumer with logarithmic preferences equals the inverse of the return on the market index.⁴ Since $E_t[\lambda_{t^-}^{t+\tau}/b(r(t), \tau)] = 1$, it follows that:

$$0 = E_{t^-}[\frac{I(t^-)}{I(t)}(c_i^o(t, k, \tau) - c(s_i(t), r(t), k, \tau))]. \quad (7)$$

Equation (7) is a necessary condition for the Cox-Ingersoll-Ross equilibrium to remain unaltered after the introduction of the batch call option market with execution price uncertainty.

Equation (7) will be the focus of the methodological and empirical part of this paper. It requires the *pricing error*, defined as the difference between the option clearing price and the option price (that would obtain in a fully synchronous option market), divided by the return on the market, to be orthogonal to information which the representative investor observes when submitting his/her order to the option market. Equation (7) is related to the null hypothesis investigated in most of the empirical option-pricing literature.⁵ The null hypothesis in the traditional literature requires the pricing error to be zero on average, and uncorrelated with instruments such as the exercise price and the time to maturity. In contrast, (7) requires the pricing error *divided by the return on the market* to be zero on average, and uncorrelated to the exercise price and the time to maturity. Moreover, it extends the instrument list to anything in the information set of the representative investor at the time of order submission. Naturally, the traditional null hypothesis is *ad hoc*, in contrast to (7), which is based on a general equilibrium model.

⁴See also Rubinstein [1976].

⁵See Galai[1983], or Hull [1989], section 12.11, for an overview of the empirical literature.

3 The Methodology

Before discussing the procedure for estimating the parameters of the model and testing (7), we shall briefly describe the nature of the data set to be used in the empirical section and, subsequently, rewrite the option pricing formula (6) accordingly.

We will work with a panel data set of call options written on nondividend-paying common stock.⁶ The first three transactions of call options written on the same underlying asset are collected at, for example, fifteen-minute intervals, thus assuming that the time from submission to execution ($t - t^-$) equals fifteen minutes. As before, let $c_i^o(t, k, \tau)$ denote a call option transaction price. We also collect the synchronous stock price quote ($s_i(t)$) and the synchronous stock index quote ($I(t)$). The continuous-time interest rate, $r(t)$, is obtained from synchronous quotes of a 90-day Treasury bill futures contract by inverting the futures pricing formula that obtains in the present economy (given in Cox, Ingersoll and Ross [1981]). $h(t, \tau_h)$ will denote the time t price for a 90-day Treasury-bill futures contract which matures at time $t + \tau_h$.

Having described the nature of the data set, we shall now rewrite the option pricing formula (6) accordingly. First, we shall subscript k and τ to reflect the fact that the first three options collected every fifteen minutes will generally have a different exercise price and time-to-maturity. Similarly, we shall make k and I time-dependent. Hence, we shall write: $k_j(t), \tau_j(t), t = 1 \dots T; j = 1, \dots, 3$. (T is the sample size in time.) The clearing price of option j at time t will be written $c_i^o(t, k_j(t), \tau_j(t))$. Second, we shall write the theoretical call price explicitly as a function of the vector of parameters θ , where $\theta = [\sigma_i' \sigma_x \rho_i \alpha \beta']'$. Let θ^* denote the value of θ for which equation (7) holds. Third, we shall write $r(t)$ explicitly as a function of $h(t, \tau_h(t))$, the price of a 90-day Treasury-bill futures contract with maturity $\tau_h(t)$. (τ_h is made time dependent to reflect that synchronous futures quotes do not necessarily refer to the same contract.) Expressing time in number of years,⁷ Cox, Ingersoll and Ross [1981, p. 344] implies:

$$r(h(t, \tau_h(t)), \tau_h(t); \theta) = \frac{-\ln h(t, \tau_h(t)) + \ln \{ A^h [\frac{\eta(\tau_h(t))}{B^h + \eta(\tau_h(t))}]^{\frac{2\alpha\beta'}{\sigma_x^2}} \}}{C^h(\tau_h(t))} \quad (8)$$

where,

$$A^h = \left[\frac{2\gamma e^{0.125(\alpha+\gamma)}}{(\alpha + \gamma)(e^{0.25\gamma} - 1) + 2\gamma} \right]^{\frac{2\alpha\beta'}{\sigma_x^2}},$$

⁶By focusing on nondividend-paying commonstock we avoid having to distinguish between American and European options. The former can always be exercised. The latter can be exercised only at maturity. Most traded options, including the ones investigated in the empirical section of this paper, are American.

⁷Hence, the 90-day Treasury bill has 0.25 years until maturity.

$$\begin{aligned}
B^h &= \frac{2(e^{0.25\gamma} - 1)}{(\alpha + \gamma)(e^{0.25\gamma} - 1) + 2\gamma}, \\
C^h(\tau_h(t)) &= \frac{\eta(\tau_h(t))B^he^{-\alpha\tau_h(t)}}{B^h + \eta(\tau_h(t))}, \\
\gamma &= (\alpha^2 + 2\sigma_x^2)^{\frac{1}{2}}, \\
\eta(\tau_h(t)) &= \frac{2\alpha}{\sigma_x^2(1 - e^{-\alpha\tau_h(t)})}
\end{aligned}$$

Notice that we have imposed, without loss of generality, the local expectations hypothesis. Fourth, in order to avoid the integral over the interest rate process $(\int_t^{t+\tau} r(u)du)$ under the expectations operator in (6), deflate all prices by the price of a $\tau_j(t)$ period zero coupon bond, $b(r(h(t, \tau_h(t)), \tau_h(t); \theta), \tau_j(t))$. From Huang [1985], we can write the deflated call price as an expectation over the risk-neutral process of the *deflated* price of the underlying asset. An analytical expression for the deflator exists in the present economy (see Cox, Ingersoll and Ross [1985b]):

$$\begin{aligned}
b(r(h(t, \tau_h(t)), \tau_h(t); \theta), \tau_j(t)) &= \\
A(\tau_j(t))e^{-B(\tau_j(t))r(h(t, \tau_h(t)), \tau_h(t); \theta)}, & \tag{9}
\end{aligned}$$

$$\begin{aligned}
A(\tau_j(t)) &= \left[\frac{2\gamma e^{(\alpha+\gamma)\tau_j(t)/2}}{(\alpha + \gamma)(e^{\gamma\tau_j(t)} - 1) + 2\gamma} \right]^{\frac{2\alpha\beta'}{\sigma_x^2}}, \\
B(\tau_j(t)) &= \frac{2(e^{\gamma\tau_j(t)} - 1)}{(\alpha + \gamma)(e^{\gamma\tau_j(t)} - 1) + 2\gamma}, \\
\gamma &= (\alpha^2 + 2\sigma_x^2)^{\frac{1}{2}}
\end{aligned}$$

Again we have imposed, without loss of generality, the local expectations hypothesis.

Consequently,

$$\begin{aligned}
c(s_i(t), r(t), k_j(t), \tau_j(t); \theta) &= \\
k_j(t)c'(\frac{s_i(t)}{k_j(t)}, h(t, \tau_j(t)), \tau_j(t), \tau_h(t); \theta), & \tag{10}
\end{aligned}$$

where

$$\begin{aligned}
c'(\frac{s_i(t)}{k_j(t)}, h(t, \tau_h(t)), \tau_j(t), \tau_h(t); \theta) = \\
A(\tau_j(t))e^{-B(\tau_j(t))r(h(t, \tau_h(t)), \tau_h(t); \theta)} \\
E_{h(t, \tau_h(t)), f'(t)}[\max(f'(t + \tau_j(t)) - 1, 0)],
\end{aligned} \tag{11}$$

with

$$f'(t + \tau_j(t)) = e^{F'(t + \tau_j(t))}$$

$$\begin{aligned}
dF'(u) = \\
-\frac{1}{2}(\sigma_i'^2 + 2\sigma_i' B(\tau_j(t) - (u - t))\sigma_x \rho_i + B(\tau_j(t) - (u - t))^2 \sigma_x^2) \\
\cdot (R(u))^2 du + \sigma_i' R(u) d\tilde{z}_i(u) + B(\tau_j(t) - (u - t))\sigma_x R(u) dz_x(u), \\
u \in [t, t + \tau_j(t)],
\end{aligned}$$

$$\begin{aligned}
dR(u) = \frac{1}{2} \frac{1}{R} [\alpha(\beta' - R^2) - \frac{1}{4} \sigma_x^2] dt + \frac{1}{2} \sigma_x dz_x, \\
u \in [t, t + \tau_j(t)],
\end{aligned}$$

$$F'(t) = \ln f'(t),$$

$$f'(t) = \frac{s_i(t)}{A(\tau_j(t))e^{-B(\tau_j(t))r(h(t, \tau_h(t)), \tau_h(t); \theta)} k_j(t)}$$

$$R(t) = (r(h(t, \tau_h(t)), \tau_h(t); \theta))^{\frac{1}{2}}$$

Again we have used the homogeneity of the risk-neutral process of the deflated price of the underlying asset to express the value of a call as a function of prices normalized by the exercise price (equation (10)). Also, the processes over which the expectation in (11) is defined are written in terms of $F'(t)$ and $R(t)$, the logarithm of the deflated price of the underlying asset, normalized by the exercise price, and the square root of the interest rate, respectively. This will prove useful in verifying the asymptotics of the estimation procedure, which we now turn to.

The null hypothesis to be tested comes conveniently in the form of *moment conditions* (equation (7)). After selecting instruments one can estimate the parameters of the model (θ) by minimizing a quadratic form in the *sample* version of the moment conditions (estimation by analogy). The distance between the minimum of the quadratic form and zero provides a test of the null hypothesis. This is the idea behind Hansen's GMM [1982].

One should, however, carefully check the conditions under which the GMM estimator is consistent, and converges to a normally distributed random vector when multiplied by the square root of the sample size. Most importantly, the random variable

under the expectations operator in (7) and the instruments one selects from the information set at t^- have to be stationary and ergodic. Given stationarity and ergodicity of the return on the market and of the instruments, the pricing error $(c_i^o(t, k_j(t), \tau_j(t)) - k_j(t)c'(\frac{s_i(t)}{k_j(t)}, h(t, \tau_h(t)), \tau_j(t), \tau_h(t); \theta))$ in the notation of the present section) has to be stationary and ergodic. This requirement may not hold because stock and call prices behave very much like random walks. One can, however, divide the pricing error by the exercise price, $k_j(t)$, without affecting the moment conditions (7). The redefined pricing error should be stationary and ergodic because options exchanges always reset exercise prices with reference to the going price of the underlying asset when introducing new contracts. Consequently, the null hypothesis becomes (setting $t^- = t - 1$):

$$0 = E_{t-1} \left[\frac{I(t-1)}{I(t)} \left(\frac{c_i^o(t, k_j(t), \tau_j(t))}{k_j(t)} - c' \left(\frac{s_i(t)}{k_j(t)}, h(t, \tau_h(t)), \tau_j(t), \tau_h(t); \theta^* \right) \right) \right] \quad (12)$$

Consider the following instruments (fifteen in total): $y_{(j-1)5+1}(t) = 1$; $y_{(j-1)5+2}(t) = s_i(t-1)/k_j(t)$; $y_{(j-1)5+3}(t) = \tau_j(t)$; $y_{(j-1)5+4}(t) = h(t-1, \tau_h(t-1))$; $y_{(j-1)5+5}(t) = c_i^o(t-1, k_j(t-1); \tau_j(t-1)) / k_j(t-1)$ ($j = 1, 2, 3$).⁸ These instruments can be assumed to be stationary and ergodic. Use them in conjunction with (12) and the law of iterated expectations to generate fifteen moment conditions:

$$0 = E \left[\frac{I(t-1)}{I(t)} \left(\frac{c_i^o(t, k_j(t), \tau_j(t))}{k_j(t)} - c' \left(\frac{s_i(t)}{k_j(t)}, h(t, \tau_h(t)), \tau_j(t), \tau_h(t); \theta^* \right) \right) y_q(t) \right], \quad (13)$$

$q = 1, \dots, 15$, $j = 1$ for $q = 1, \dots, 5$; $j = 2$ for $q = 6, \dots, 10$; $j = 3$ for $q = 11, \dots, 15$. The sample version of (13) can then be used to generate a GMM estimator of θ^* . Provided some additional assumptions are satisfied, the GMM estimator will be consistent and asymptotically normally distributed (Hansen [1982]). Since there are fifteen moment conditions and five parameters to be estimated, the minimum of the quadratic form in the sample moment conditions times the sample size provides a χ^2 statistic with ten degrees of freedom with which to test the null hypothesis in (13).

Unfortunately, GMM estimation is not possible in the absence of an analytical expression for the price of a call option $(c'(\frac{s_i}{k_j(t)}, h(t, \tau_h(t)), \tau_j(t), \tau_h(t); \theta))$. But this price is defined as an expectation over the continuous-time processes of $R(u)$ and $F'(u)$ ($u \in [t, t + \tau_j(t)]$; see equation (11)). Consequently, as in Boyle [1977] we propose substituting a *Monte Carlo* estimator for the analytically unknown theoretical call price. For each observation in the sample we shall estimate the corresponding theoretical call price by

⁸The first instrument should capture the average mispricing. The second, third and fourth ones should capture the exercise-price, time-to-maturity and interest-rate biases, respectively. The last instrument should pick up predictability from past call prices.

drawing N independent (across draws and across observations) realizations from the conditional distribution of $R(t + \tau_j(t))$ and $F'(t + \tau_j(t))$ given $R(t)$ and $F'(t)$. The substitution of simulation estimates for analytically unknown expectations in moment conditions and the subsequent GMM estimation has become known as Method of Simulated Moments (MSM) (See McFadden [1989] and Pakes and Pollard [1989]). As in McFadden [1989], the fact that the simulation estimates (and the corresponding expectations) enter the moment conditions *linearly* can be exploited to show consistency and asymptotic normality of the MSM estimator of θ^* even if the number of simulations per observation (N) is kept fixed as the sample size (T) increases. The asymptotic variance-covariance matrix of the parameter estimates, and hence, the power, will depend on N . Intuitively, by increasing N , the “size” of the simulation error (defined as the simulated call price minus the theoretical call price) decreases relative to the “size” of the pricing error (defined as the observed call price minus the theoretical call price). As in Boyle [1977], we shall keep the “size” of the simulation error at a minimum by means of variance reduction, using a closely related model for which an analytical solution does exist, namely, the Black-Scholes [1973] model⁹

The asymptotics of the MSM estimator, in particular its asymptotic normality, does not follow directly from Hansen [1982], because of the nondifferentiability of the payoff on the call option when the call ends up “at the money” at maturity (i.e., $f'(t + \tau_j(t)) = 1$). Using convex analysis, the Appendix investigates the asymptotic properties of the MSM estimator and restores Hansen’s results.

The MSM estimator needs to be refined further, however, because of lack of analytical knowledge of the conditional distribution of $R(t + \tau_j(t))$ and $F'(t + \tau_j(t))$ given $R(t)$ and $F'(t)$, from which random draws are taken. Nevertheless, we shall be able to obtain N simulations of $R(t + \tau_j(t))$ and $F'(t + \tau_j(t))$ by *discretizing* their processes using a simple Euler scheme. We shall discretize over M intervals of length $\tau_j(t)/M$. In particular, we shall take $T \times N \times 3$ independent pairs of Brownian motions on the interval $[0, 1]$, denoted $(\tilde{z}_i(t, j, n), z_x(t, j, n))$, $t = 1, \dots, T$; $j = 1, 2, 3$, $n = 1, \dots, N$, with correlation ρ_i , and generate estimates $R_n^{(M)}(t + \tau_j(t))$ and $F_n^{(M)}(t + \tau_j(t))$ ($n = 1, \dots, N$) from the following stochastic difference equations:

$$\begin{aligned} F_n^{(M)}(t + l \frac{\tau_j(t)}{M}) = & \\ & F_n^{(M)}(t + (l-1) \frac{\tau_j(t)}{M}) - (\sigma_i'^2 + 2\sigma_i' B(\tau_j(t)(1 - \frac{l-1}{M}))\sigma_x \rho_i) \\ & + B(\tau_j(t)(1 - \frac{l-1}{M}))^2 \sigma_x^2 (R_n^{(M)}(t + (l-1) \frac{\tau_j(t)}{M}))^2 \frac{\tau_j(t)}{M} \\ & + \sigma_i' R_n^{(M)}(t + (l-1) \frac{\tau_j(t)}{M}) (\tilde{z}_i(t, j, n)(\frac{l}{M}) - \tilde{z}_i(t, j, n)(\frac{l-1}{m})) \end{aligned}$$

⁹When the correlation between the stock price process and the interest rate process (ρ_i) is zero, the call pricing model of this paper and the Black-Scholes model generate almost identical prices. See Bailey and Stulz [1989] for examples.

$$\begin{aligned}
& + B(\tau_j(t)(1 - \frac{l-1}{M}))\sigma_x R_n^{(M)}(t + (l-1)\frac{\tau_j(t)}{M}) \\
& \cdot (z_x(t, j, n)(\frac{l}{M}) - z_x(t, j, n)(\frac{l-1}{M})),
\end{aligned} \tag{14}$$

$$\begin{aligned}
R_n^{(M)}(t + l\frac{\tau_j(t)}{M}) = & R_n^{(M)}(t + (l-1)\frac{\tau_j(t)}{M}) + \{\frac{1}{2}[\alpha(\beta' - (R_n^{(M)}(t + (l-1)\frac{\tau_j(t)}{M}))^2) \\
& - \frac{1}{4}\sigma_x^2]/R_n^{(M)}(t + (l-1)\frac{\tau_j(t)}{M})\}\frac{\tau_j(t)}{M} \\
& + \frac{1}{2}\sigma_x(z_x(t, j, n)(\frac{l}{M}) - z_x(t, j, n)(\frac{l-1}{M})),
\end{aligned} \tag{15}$$

$$l = 1, \dots, m,$$

$$n = 1, \dots, N,$$

$$F_n^{(M)}(t) = F'(t),$$

$$R_n^{(M)}(t) = R(t).$$

The discretization estimates $R_n^{(M)}(t + \tau_j(t))$ and $F_n^{(M)}(t + \tau_j(t))$ of $R(t + \tau_j(t))$ and $F'(t + \tau_j(t))$ are biased, however, in the sense that:

$$\begin{aligned}
& E_{h(t, \tau_j(t)), f'(t)}[\frac{1}{N} \sum_{n=1}^N \max(f_n^{(M)}(t + \tau_j(t)) - 1, 0)] \neq \\
& E_{h(t, \tau_h(t)), f'(t)}[\max(f'(t + \tau_j(t)) - 1, 0)],
\end{aligned}$$

where $f_n^{(M)}(t + \tau_j(t)) = e^{F_n^{(M)}(t + \tau_j(t))}$. Nevertheless, the bias can be shown to disappear as $M \rightarrow \infty$. Consequently, in order for the asymptotics of the MSM estimator of θ^* to remain valid after substituting discretization estimates for random draws from the conditional distribution of $R(t + \tau_j(t))$ and $F'(t + \tau_j(t))$, M has to increase with the sample size (T). In particular, the Appendix shows that $T/M(T)$ has to converge to 0 as $T \rightarrow \infty$.

While the bias thus disappears asymptotically, we are still left with a finite sample bias. We shall minimize it using Talay and Tubaro's [1989] extension of the Romberg

interpolation to approximations of stochastic differential equations.¹⁰ In short, the empirical investigation we now turn to will show results from MSM estimation of θ^* , where discretization simulation estimates substitute the analytically unknown theoretical call prices. The discretization simulation estimates are subject to variance reduction and Romberg interpolation in order to maximize the power and minimize small sample biases.

4 Empirical Results

We shall first describe the dataset in more detail, followed by a discussion of how we determined M , the number of intervals in the discretization, and N , the number of simulations per observation. Finally, the estimation results are reported and interpreted.

We collected transaction prices of call options on nondividend-paying common stock from the Berkeley Options Data Base for the period January through June 1986. Three call price series and the (one) corresponding stock price series were generated for common stock of three companies (Bank of America, Federal Express and Bethlehem Steel)¹¹, by picking transactions closest after each fifteen-minute, thirty-minute and sixty-minute mark. Observations had, however, to be deleted because less than three call transactions took place in the corresponding interval. This problem of “thin trading” was acute for Federal Express and Bethlehem Steel. Consequently, we constructed an alternative dataset for those two companies by picking only two call transactions in each fifteen-minute, thirty-minute and sixty-minute interval. While the number of parameters stayed the same, the number of moment conditions dropped to ten. Consequently, there were only five degrees of freedom.¹² Matching 90-day Treasury bill futures quotes were collected, closest after each fifteen-minute, thirty-minute and sixty-minute mark. They were obtained from Chicago’s Mercantile Exchange. Likewise, index (S & P 100) quotes closest after each fifteen-minute, thirty-minute and sixty-minute mark, were obtained from the Berkeley Options Data Base.¹³ From the first observation of the day, only the index quote was used, to calculate the return on the index. Consequently, our dataset did not

¹⁰As Talay and Tubaro [1989] show, the Euler discretization scheme we employ, together with Romberg interpolation, often outperforms higher-order discretization schemes.

¹¹These companies did not pay dividends over the life of the options that were written on them in the first half of 1986.

¹²The lack of transactions for Federal Express and Bethlehem Steel implies an additional source of risk beyond execution price uncertainty, namely *execution time uncertainty*: when a bid or offer is posted in the options market, it may take some time before it is matched. Our model obviously ignores transaction time uncertainty, because it assumes that trades will be executed at the time that they were ordered to be executed. In other words, our model explains pricing only in very active markets. We therefore decided to exclude observations with less than three or two transactions from the dataset (which correspond to periods of low activity), rather than increasing the length of the time intervals to the point that each observation included three transactions.

¹³The intense activity in the Treasury bill futures market and the index option market meant that futures and index quotes could generally be obtained after each mark.

include overnight intervals, but only daytime intervals.^{14 15}

Since only the asymptotic properties of the Method of Simulated Moments are known, we focused on the longest time series for each company. In other words, we took, for Bank of America, the dataset consisting of observations on three call transactions following each fifteen-minute mark. For Federal Express and Bethlehem Steel, we took the datasets of observations on *two* call transactions following each fifteen-minute interval.

Table 1 provides descriptive statistics of the three time series. In addition to means, minima and maxima of call, stock and exercise prices, time-to-maturities, futures quotes and index returns, it also displays summary statistics on the extent of nonsynchrony in the dataset, such as the average time between each fifteen-minute mark and the first call option transaction.

In order to determine suitable values for M , the number of intervals used in the discretization, and N , the number of simulations per observation, we proceeded as follows. First, we took the average stock price and average time-to-maturity for Bank of America, and estimated theoretical call option values for different exercise prices, using several discretization sizes. In order to be able to focus solely on discretization-induced biases, we set N , the number of simulations, equal to 10000. The continuous-time stock price and interest rate processes were approximated over 2, 4, 8,..., 256 intervals. Reasonable values were chosen for the stock price volatility parameter (σ'_i), the adjustment speed (α) and the long-run interest rate (β'), but the interest rate volatility parameter (σ_x) was set rather high, in order to exacerbate any discretization-induced biases. Table 2 reports the results for two different values of the correlation between the stock price and the interest rate (ρ_i). As can be verified, there are hardly any biases. Consequently, for the purpose of model estimation, we choose a relatively small value for M , namely 8.¹⁶

Second, using the same parameter values but a zero correlation coefficient, we calculated the standard deviation of the pricing error and the simulation error (both normalized by the exercise price) in each dataset. The results are reported in Table 3. The standard deviation of the simulation error obtained with only one simulation per observation is one-half that of the pricing error.¹⁷ Consequently, we expected our tests to be powerful with just a few simulations per observation. In the estimation, we decided to

¹⁴Overnight intervals create difficulties because of (1) different time-series properties (in particular, they create unconditional heteroscedasticity), and (2) dividends being paid on component stock of the index.

¹⁵There is another complication that we ignore. The model we wish to test is expressed in real terms, yet the data come in nominal terms. We assume that this difference is immaterial. (Besides, how would one proceed to adjust nominal transaction data observed over at most one-hour periods for inflation?)

¹⁶As in Bailey and Stulz [1989], the call prices are estimated to be higher than the Black-Scholes prices for a positive correlation between the stock price and the interest rate. The opposite (not reported) occurred for negative correlation. Also, the percentage mispricing of the Black-Scholes model increases with the exercise price.

¹⁷The lower standard deviation of the simulation error for Federal Express can be explained by the substantially shorter average time-to-maturity. See Table 1.

use ten simulations per observation (i.e., $N = 10$).¹⁸

The estimation of the full model turned out to be a difficult exercise. For the three companies, the search algorithms (GQOPT's DFP and simplex methods) converged to a corner point, with σ_x , the interest rate volatility, equal to zero.¹⁹ We tried an alternative route: we fixed σ_x at a reasonably small value (0.05) and re-ran the estimations. Unfortunately, other parameters moved to their respective corner points.²⁰ A grid search confirmed that the optimum was characterized by a zero interest rate volatility.

Consequently, it appeared to be impossible to fit the Cox-Ingersoll-Ross model to the data for a nonzero volatility of the interest rate. We attempted to fit the Cox-Ingersoll-Ross formula for a zero interest rate volatility and speed of adjustment. This model is *internally inconsistent*: while the interest rate implicit in the Treasury bill futures quotes changes over time and affects the volatility of the common stock, it is assumed to remain constant over the remainder of the life of the option and the futures contract. Yet, it provides a reasonable fit, as can be verified in the first panel of Table 4. The stock price volatility parameter is estimated very precisely and both the mean and the standard deviation of the pricing error are very low. Nevertheless, the model is convincingly rejected, which means that there is substantial predictability in the pricing errors (divided by the return on the market).

We also estimated the Cox-Ingersoll-Ross model by fixing the interest rate volatility at a very low value (0.001), and setting the adjustment speed and the correlation equal to 1 and 0, respectively. This model is internally consistent, but, obviously, does not differ much from the previous one. Consequently, as can be verified from the second panel of Table 4, the estimation results are similar.

Finally, we estimated the Black-Scholes model, by imposing that the stock price volatility (σ_i') and the interest rate (R) be constant over time. Rubinstein's [1976] results can be used to show that this model is internally consistent: the pricing error should be orthogonal to the inverse return on the market. As with the full Cox-Ingersoll-Ross model, the estimation turned out to be difficult. The estimates of both the stock price volatility and the interest rate converged to zero. At that point, however, the pricing errors are still substantial, as can be verified from the third panel in Table 4. Yet, the average pricing error divided by the market return is smallest and least predictable for a zero interest rate and volatility.

In an attempt to explain these rejections, we ran Ordinary Least Squares regressions of the pricing error divided by the market return onto the instruments. The pricing

¹⁸When calculating the weighting matrix in the second step of the MSM, however, we set $N = 40$. This should minimize biases in the χ^2 -test of the over-identifying moment restrictions.

¹⁹The convergence was extremely slow, however, because ρ_i , the correlation between the interest rate and the stock price, becomes unidentified as $\sigma_x \rightarrow 0$.

²⁰The corner points are: $\rho_i = 1, -1$; $2\alpha\beta' = \sigma_x^2$ (the latter is the corner point beyond which the discretization procedure does not converge; see (A5) in the Appendix).

errors were calculated at the optimum for the first model in table 4 (the Cox-Ingersoll-Ross model with $\sigma_x = 0$ and $\alpha = 0$). Table 5 reports the results. A large part of the rejections is clearly due to the interest rate: the pricing error divided by the market return is systematically correlated with the lagged Treasury bill futures quote. There is some, albeit less systematic, evidence of the type of pricing biases that have usually been associated with option pricing (exercise-price and time-to-maturity biases).

This lead us to *conjecture* that the rejections were not caused by the relatively short time intervals (the pricing errors divided by the return on the market have to be orthogonal to information that is often less than fifteen minutes old). Additional investigation seemed to confirm this: when we attempted to lag the instruments one more period, the estimation results did not alter substantially. Since a significant part of the rejections can be attributed to the interest rate, we are lead to question the appropriateness of the way stock price volatility is modeled in the Cox-Ingersoll-Ross model. In particular, the data seem to object to the one-to-one link between stock price volatility and the interest rate. The data do agree with the hypothesis that stock price volatility changes over time (evidenced by the rejection of the Black-Scholes model). Moreover, they do agree with the hypothesis that such changes are related to changes the interest rate (evidenced by the relative success of the Cox-Ingersoll-Ross model with $\sigma_x = 0$ and $\alpha = 0$). Yet, they reject the notion that volatility changes can completely be captured by interest rate changes.

The conjecture that the rejections were not caused by the relatively short time intervals but by the inappropriate modeling of the stock price volatility received additional support from results of the following exercise. We calculated the χ^2 statistics for the Black-Scholes model using the volatility implied from the preceding pricing error²¹ and the interest rate implied from the Treasury bill futures quote (imposing a flat term structure). Like the first model, this one is also internally inconsistent. Yet, the results, displayed in the fourth panel of Table 4, are interesting: the χ^2 s improve substantially over those for the other models. As a matter of fact, we now fail to reject for Bethlehem Steel.

5 Conclusion

In this paper, batch call option markets with execution price uncertainty were introduced in the economy of Cox, Ingersoll and Ross [1985a]. Necessary conditions on the nature of the execution price uncertainty were established for the original equilibrium to be unaffected. The Method of Simulated Moments procedure was extended to test these conditions. Using a dataset of call option transactions from the Chicago Board Options Exchange, the paper found reliable evidence against these conditions. Further analysis lead to the conjecture that the rejections should be attributed to the fact that the Cox-

²¹The previous pricing error could often not be set equal to zero at a positive volatility. In that case, the implied volatility was not recalculated.

Ingersoll-Ross model relates changes in stock price volatility exclusively to changes in the interest rate.

Appendix²²

We shall now state the precise conditions under which the MSM estimator introduced in this paper is consistent and converges to a normally distributed random vector. The MSM estimator concerns a unique (5×1) parameter vector θ^* in the following moment conditions:

$$E[g(x(t); \theta^*)] = 0, \quad (16)$$

where:

$$x(t)' = \left[\frac{I(t-1)}{I(t)}, \left\{ \frac{c_i^o(t, k_j(t), \tau_j(t))}{k_j(t)} \right\}_{j=1}^3, \left\{ \frac{s_i(t)}{k_j(t)} \right\}_{j=1}^3, h(t, \tau_h(t)), \{\tau_j(t)\}_{j=1}^3, \tau_h(t), \right. \\ \left. \left\{ \frac{s_i(t-1)}{k_j(t)} \right\}_{j=1}^3, h(t-1, \tau_h(t-1)), \left\{ \frac{c_i^o(t-1, k_j(t-1), \tau_j(t-1))}{k_j(t-1)} \right\}_{j=1}^3 \right]$$

and $g(x(t); \theta^*)$ is a (15×1) vector, the q th element of which equals:

$$g_q(x(t); \theta^*) = \frac{I(t-1)}{I(t)} \left(\frac{c_i^o(t, k_j(t), \tau_j(t))}{k_j(t)} - c' \left(\frac{s_i(t)}{k_j(t)}, h(t, \tau_h(t)), \tau_j(t), \tau_h(t), \theta^* \right) y_q(t), \right.$$

where $j = 1$ for $q = 1, \dots, 5$; $j = 2$ for $q = 6, \dots, 10$; $j = 3$ for $q = 11, \dots, 15$; and $y_1(t) = 1$; $y_2(t) = s_i(t-1)/k_j(t)$; $y_3(t) = \tau_j(t)$; $y_4(t) = h(t-1, \tau_h(t-1))$; $y_5(t) = c_i^o(t-1, k_j(t-1), \tau_j(t-1))/k_j(t-1)$. The MSM estimator $\theta(T)$ of θ^* is defined as the argument that minimizes a quadratic form in the sample moment conditions corresponding to (16), constructed from a sample of size T , and after substituting a simulation estimator for $c'(\frac{s_i(t)}{k_j(t)}, h(t, \tau_h(t)), \tau_j(t), \tau_h(t); \theta)$. The latter equals:

²²This Appendix applies (and corrects) results previously explained in a working paper by the first author, entitled, "The Asymptotic Normality of Method of Simulated Moments Estimators of Option Pricing Models".

$$c'^{(M(T),N)}(\frac{s_i(t)}{k_j(t)}, h(t, \tau_h(t)), w(t), \tau_j(t), \tau_h(t); \theta) =$$

$$A(\tau_j(t))e^{-B(\tau_j(t))r(h(t, \tau_h(t)), \tau_h(t); \theta)} \frac{1}{N} \sum_{n=1}^N \max (f_n'^{(M(T))}(t + \tau_j(t)) - 1, 0),$$

where N indicates the number of simulations, and $M(T)$ (which increases with T) the number of intervals over which the process of f'_n is discretized. The formulae for the discretization are given in equations (14) and (15) of the paper. $w(t)$ is a vector of standard Brownian motions on $[0, 1]$ which are used to generate the $T \times N \times 3$ simulations.²³

$$w(t)' = [\{\{\tilde{z}_i(t, j, n)\}_{n=1}^N\}_{j=1}^3, \{\{z_x(t, j, n)\}_{n=1}^N\}_{j=1}^3],$$

where $\tilde{z}_i(t, j, n)$ and $z_x(t, j, n)$ are standard Brownian motions on $[0, 1]$, with correlation ρ_i , but uncorrelated across t, j and n . Consequently, the sample moment conditions corresponding to (16) are:

$$\frac{1}{T} \sum_{t=1}^T g^{(M(T),N)}(x(t), w(t); \theta) \tag{17}$$

where $g^{(M(T),N)}(x(t), w(t); \theta)$ is a (15×1) vector, the q th element of which equals:

$$g_q^{(M(T),N)}(x(t), w(t); \theta) =$$

$$\frac{I(t-1)}{I(t)} \left(\frac{c_i^o(t, k_j(t), \tau_j(t))}{k_j(t)} - c'^{(M(T),N)}(\frac{s_i(t)}{k_j(t)}, h(t, \tau_h(t)), w(t), \tau_j(t), \right.$$

$$\left. \tau_h(t); \theta) \right) y_q(t)$$

In order to define the criterion function of the MSM estimator, let $D(T)$ be a (15×15) symmetric, positive definite matrix that converges in probability to the symmetric, positive definite D^* as $T \rightarrow \infty$. The MSM estimator $\theta(T)$ is defined as:

$$\theta(T) = \operatorname{argmin} \left(\frac{1}{T} \sum_{t=1}^T g^{(M(T),N)}(x(t), w(t); \theta)' \right)$$

$$D(T) \left(\frac{1}{T} \sum_{t=1}^T g^{(M(T),N)}(x(t), w(t), \theta) \right) \tag{18}$$

²³There are N simulations per call; three calls per observations; and T observations.

As far as the asymptotics are concerned, there are two major differences between the above MSM estimator and Hansen's [1982] GMM estimator. First, $g^{(M(T),N)}(x(t), w(t); \theta)$, while continuous, is nondifferentiable in θ whenever $f_n^{(M(T))}(t + \tau_j(t)) = 1$, i.e., whenever a simulation results in an "at the money" call option at maturity. Since there are N simulations per option, and three options per observation, there are at most $3 \times N$ nondifferentiabilities ("kinks") in $g^{(M(T),N)}(x(t), w(t); \theta)$. Second, because of the biases in the simulation estimator of the call price formula when the process of the underlying asset is discretely approximated, $E[g^{(M(T),N)}(x(t), w(t); \theta)]$ will not equal zero at $\theta = \theta^*$.

The nondifferentiabilities do not complicate the proof of consistency of $\theta(T)$, yet they do affect the proof of asymptotic normality of $\sqrt{T}(\theta(T) - \theta^*)$, because its arguments have traditionally been based on a Taylor expansion of the sample moment conditions about θ^* . The ability to appeal to a Taylor expansion argument requires that $g^{(M(T),N)}(x(t), w(t); \theta)$ be differentiable with probability 1 in a *neighborhood* of θ^* , a condition that is clearly violated. The nondifferentiabilities, however, are caused by the function $\max(\cdot, 0)$, which is convex. Consequently, one can appeal to convex analysis to generate a Taylor-like expansion, as follows. Let $F(y)$ be a scalar convex function with a finite number of nondifferentiabilities. Convexity implies:

$$\begin{aligned} F(y) - F(y^*) &\geq \frac{d}{dy} F(y^*)(y - y^*), \\ F(y^*) - F(y) &\geq \frac{d}{dy} F(y)(y^* - y), \end{aligned} \tag{19}$$

where $\frac{d}{dy} F(y)$ denotes the left derivative of F with respect to y , to avoid ambiguities at the points of nondifferentiability. Consequently:

$$F(y) = F(y^*) + (\lambda \frac{d}{dy} F(y) + (1 - \lambda) \frac{d}{dy} F(y^*))(y - y^*), \tag{20}$$

with $\lambda \in [0, 1]$. The expansion in (20) will be applied to $F(\cdot) = \max(\cdot, 0)$ in the proof of asymptotic normality of $\theta(T)$.

If we want to insist that $E[g^{(M(T),N)}(x(t), w(t); \theta)]$ equal zero at $\theta = \theta^*$ *asymptotically* (which we have to in order for the estimator based on the corresponding sample moment conditions to be consistent), the bias in the discretization simulation estimator have to disappear as $T \rightarrow \infty$. Consequently, M , the number of intervals over which the process of the underlying asset is discretized, ought to increase with T . Moreover, for the bias not to affect the asymptotic variance-covariance matrix of $\theta(T)$, M ought to increase at an appropriate rate. We shall determine this shortly.

Let us first introduce additional notation.

$$G(x(t); \theta) = \frac{d}{d\theta'} g(x(t); \theta), \quad (21)$$

i.e., $G(x(t); \theta)$ is a (15×5) matrix, the rows of which are the gradients of $g(x(t); \theta)$ with respect to θ . Similarly:

$$G^{(M(T), N)}(x(t), w(t); \theta) = \frac{d}{d\theta'} g^{(M(T), N)}(x(t), w(t); \theta), \quad (22)$$

with rows

$$\begin{aligned} \frac{d}{d\theta'} g_q^{(M(T), N)}(x(t), w(t); \theta) = \\ -\frac{1}{N} \sum_{n=1}^N 1_{\{f_n'^{(M(T))}(t+\tau_j(t)) > 1\}} \frac{I(t-1)}{I(t)} y_q(t) \\ \cdot \frac{d}{d\theta'} \{A(\tau_j(t)) e^{-B(\tau_j(t))r(h(t, \tau_h(t)); \theta)} (f_n'^{(M(T))}(t + \tau_j(t)) - 1)\}, \end{aligned}$$

where $1_{\{f_n'^{(M(T))}(t+\tau_j(t)) > 1\}} = 1$ if $f_n'^{(M(T))}(t + \tau_j(t)) > 1, 0$ otherwise.

Next, expand $\frac{1}{T} \sum_{t=1}^T g^{(M(T), N)}(x(t), w(t); \theta)$ about θ^* :

$$\begin{aligned} \frac{1}{T} \sum_{t=1}^T g_q^{(M(T), N)}(x(t), w(t); \theta) = \\ \frac{1}{T} \sum_{t=1}^T g_q^{(M(T), N)}(x(t), w(t); \theta^*) + \left[-\frac{1}{T \cdot N} \sum_{t=1}^T \sum_{n=1}^N \tilde{\lambda}_{qt n} \frac{I(t-1)}{I(t)} y_q(t) \right. \\ \cdot \frac{d}{d\theta'} \{A(\tau_j(t)) e^{-B(\tau_j(t))r(h(t, \tau_h(t)), \tau_h(t); \tilde{\theta}_q)} \\ \cdot (f_n'^{(M(T))}(t + \tau_j(t)) - 1)\} \left. \right] (\theta - \theta^*), \quad (23) \end{aligned}$$

where $f_n'^{(M(T))}(t + \tau_j(t))$ is generated at $\tilde{\theta}_q$, $\|\tilde{\theta}_q - \theta^*\| < \|\theta - \theta^*\|$, and

$$\tilde{\lambda}_{qt n} = \begin{cases} 0 & \text{if } f_n'^{(M(T))}(t + \tau_j(t)) \leq 1 \text{ at } \theta \text{ and } \theta^*, \\ 1 & \text{if } f_n'^{(M(T))}(t + \tau_j(t)) > 1 \text{ at } \theta \text{ and } \theta^*, \\ \epsilon(0, 1) & \text{if } f_n'^{(M(T))}(t + \tau_j(t)) > 1 \text{ at either } \theta \text{ or } \theta^*. \end{cases}$$

(23) is obtained by first applying (20) to $\max(\cdot, 0)$, and subsequently writing the argument of $\max(\cdot, 0)$ in a regular Taylor expansion of a continuously differentiable function. Define $\Delta^{(M(T), N)}(x(t), w(t); \theta, \theta^*)$ to be the (15×5) matrix with rows:

$$\begin{aligned} \Delta_q^{(M(T), N)}(x(t), w(t); \theta, \theta^*) = \\ -\frac{1}{N} \sum_{n=1}^N \tilde{\lambda}_{qtn} \frac{I(t-1)}{I(t)} y_q(t) \frac{d}{d\theta'} \{ A(\tau_j(t)) e^{-B(\tau_j(t))r(h(t, \tau_h(t)), \tau_h(t); \tilde{\theta}_q)} \\ \cdot (f'_n{}^{(M(T))}(t + \tau_j(t)) - 1) \}. \end{aligned}$$

In other words, the rows of $\Delta^{(M(T), N)}(x(t), w(t); \theta, \theta^*)$ are the vectors multiplying $(\theta - \theta^*)$ in (23).

We shall need the following assumptions in order for the MSM estimator to be consistent and asymptotically normal:

(A1) $x(t)$ is bounded, stationary and ergodic.²⁴

(A2) θ is defined on a compact metric space.²⁵

(A3) $\frac{T}{M(T)} \rightarrow 0$ as $T \rightarrow \infty$.

(A4) $g(x(t), \theta^*)$ is uncorrelated with information at $t-1, t-2, \dots$

(A5) $2\alpha\beta' \geq \sigma_x^2 + \epsilon$, for some $\epsilon > 0$.

(A4) is a restatement of the null hypothesis. (A5) is needed for the interest rate process to be mean reverting to the extent that the origin becomes inaccessible and, simultaneously for boundedness to result.

The following lemmas are building blocks in the proofs of consistency and asymptotic normality of $\theta(T)$.²⁶ Define:

²⁴ $\frac{I(t-1)}{I(t)}$ need not be bounded, however, for the asymptotics to hold.

²⁵In particular, the first and second elements of θ (σ'_i and σ_x) should be strictly positive. The third element (ρ_i) should be between -1 and +1.

²⁶Only outlines of the proofs are given. Details can be obtained from the author.

$$c'^{(N)}\left(\frac{s_i(t)}{k_j(t)}, h(t, \tau_j(t)), w(t), \tau_j(t), \tau_h(t); \theta\right) = \\ A(\tau_j(t))e^{-B(\tau_j(t))r(h(t, \tau_h(t)), \tau_h(t); \theta)} \frac{1}{N} \sum_{n=1}^N \max(f'_n(t + \tau_j(t)) - 1, 0),$$

where $f'_n(t + \tau_j(t))$ is generated as in equation (11) in the paper, evaluated at the Brownian motions specified in $w(t)$.

Lemma 1:

$\sqrt{T} E|c'^{(M(T),1)}\left(\frac{s_i(1)}{k_j(1)}, h(1, \tau_h(1)), w(1), \tau_j(1), \tau_h(1); \theta\right) - c'^{(1)}\left(\frac{s_i(1)}{k_j(1)}, h(1, \tau_h(1)), w(1), \tau_j(1), \tau_h(1); \theta\right)|$ converges to zero as $T \rightarrow \infty$, uniformly in θ .

PROOF (Outline)

We know²⁷ $E|F_1^{(T)}(1 + \tau_j(1)) - F_1'(1 + \tau_j(1))|^2 = O(\frac{1}{T})$, uniformly in θ , provided the drift and diffusion coefficients satisfy (uniform) Lipschitz conditions. The latter are, in the present case:

$$\begin{aligned} & \left| \frac{1}{R}(\alpha(\beta' - R^2) - \frac{1}{4}\sigma_x^2) - \frac{1}{R'}(\alpha(\beta' - R'^2) - \frac{1}{4}\sigma_x^2) \right| \leq k_1 |R - R'| \\ & |(\sigma_i^2 + 2\sigma_i' B(\tau_j(1) - (u - 1))\sigma_x \rho_i + B(\tau_j(1) - (u - 1))^2 \sigma_x^2) \\ & \quad \cdot (R^2 - R'^2)| \leq k_2 |R - R'| \\ & |\sigma_i' R - \sigma_i' R'| \leq \max\{\sigma_i'\} |R - R'| \\ & |B(\tau_j(1) - (u - 1))\sigma_x R - B(\tau_j(1) - (u - 1))\sigma_x R'| \leq k_3 |R - R'| \end{aligned}$$

for

$$\begin{aligned} k_1 &= \max_{\alpha, \beta'} \left\{ \frac{\alpha \beta'}{R_{\min}^2} + \alpha \right\}, \\ k_2 &= \max_{\alpha, \beta', \sigma_i, \sigma_x} \{ \sigma_i^2 + 2\sigma_i' B(\bar{\tau})\sigma_x \rho_i + B(\bar{\tau})^2 \sigma_x^2 \} 2R_{\max}, \\ k_3 &= \max_{\alpha, \beta', \sigma_i, \sigma_x} \{ B(\bar{\tau})\sigma_x \}, \end{aligned}$$

and R_{\min}, R_{\max} are scalars such that $P\{R < R_{\min}\} = P\{R > R_{\max}\} = 0$ and $R_{\min} > 0, R_{\max} < \infty$,²⁸ and $\bar{\tau} = \sup\{\tau_j(1)\}$. Next, using Minkowski's and Hölder's inequalities:

²⁷See, e.g., Pardoux and Talay [1985], p. 33.

²⁸Such numbers exist under (A5). This can be shown using Karatzas and Shreve [1987], p. 351.

$$\begin{aligned} \sqrt{T} E |f_1^{(M(T))}(1 + \tau_j(1)) - f_1'(1 + \tau_j(1))| \leq \\ \{(E |f_1^{(M(T))}(1 + \tau_j(1))^2)^{\frac{1}{2}} + (E |f_1'(1 + \tau_j(1))|^2)^{\frac{1}{2}}\} \\ \{TE |F_1^{(M(T))}(1 + \tau_j(1)) - F_1'(1 + \tau_j(1))|^2\}^{\frac{1}{2}}. \end{aligned}$$

The first factor on the right-hand side is bounded while the second one converges to zero, uniformly in θ , provided $\frac{T}{M(T)} \rightarrow 0$ as $T \rightarrow \infty$. The latter holds by (A3). Lemma (1) follows immediately.

Lemma 2.

$E |\frac{\partial}{\partial \theta_p} f_1^{(M(T))}(1 + \tau_j(1)) - \frac{\partial}{\partial \theta_p} f_1'(1 + \tau_j(1))|$ converges to 0 as $T \rightarrow \infty$, uniformly in θ_p , where θ_p denotes the p th element of θ .

PROOF (Outline)

As in Arnold [1973], p. 137, the processes that generate $\frac{\partial}{\partial \theta_p} f_1'(1 + \tau_j(1))$ can be obtained by writing $f_1'(t + \tau_j(1))$ as the sum of $f_1'(t)$, the integration of the drift coefficient with respect to time and the integration of the diffusion coefficient with respect to two Brownian motions, and by subsequently differentiating $f_1'(t + \tau_j(1))$ with respect to the parameters, reversing the order of differentiation and integration.²⁹ Since the processes for $\frac{\partial}{\partial \theta_p} f_1^{(M(T))}(1 + \tau_j(1))$ are obtained in a similar way (with summation replacing integration), Lemma 2 will follow if the drift and diffusion coefficients of $\frac{\partial}{\partial \theta_p} f_1'(1 + \tau_j(1))$ satisfy the (uniform) Lipschitz conditions referred to in the proof of Lemma 1. Using the fact that R will be bounded away from 0 and ∞ , it can be shown that they do.

In addition we shall need the following results:

Lemma 3.

$\sqrt{T} \{ \frac{1}{T} \sum_{t=1}^T g^{(M(T), N)}(x(t), w(t); \theta^*) \}$ converges weakly to a normally distributed random vector with mean zero and variance-covariance matrix $\Phi^{(N)}(\theta^*)$, where

$$\begin{aligned} \Phi^{(N)}(\theta^*) &= V^{pe}(\theta^*) + \frac{1}{N} V^{se}(\theta^*), \\ V^{pe}(\theta^*) &= E[g(x(t); \theta^*) g(x(t); \theta^*)'], \\ V^{se}(\theta^*) &= E[d^{(1)}(x(t), w(t); \theta^*) d^{(1)}(x(t), w(t); \theta^*)'], \end{aligned}$$

²⁹ Arnold [1973], p. 137, considers only differentiation with respect to initial values. This case can easily be extended to differentiation with respect to the parameters by considering the latter to be additional state variables with trivial processes (i.e., their values do not change over time).

with $d'^{(1)}(x(t), w(t); \theta^*)$ a (15×1) vector with elements:

$$\begin{aligned} d_q'^{(1)}(x(t), w(t); \theta^*) = & \\ & \frac{I(t-1)}{I(t)} \left(c'^{(1)}\left(\frac{s_i(t)}{k_j(t)}, h(t, \tau_h(t)), w(t), \tau_j(t), \tau_h(t); \theta^*\right) \right. \\ & \left. - c'\left(\frac{s_i(t)}{k_j(t)}, h(t, \tau_h(t)), \tau_j(t), \tau_h(t); \theta^*\right) \right) y_q(t), \end{aligned}$$

where $j = 1$ for $q = 1, \dots, 5$; $j = 2$ for $q = 6, \dots, 10$; $j = 3$ for $q = 11, \dots, 15$.

($V^{pe}(\theta^*)$ is the unconditional variance-covariance matrix of the *pricing error* multiplied by instruments and the inverse return on the market and $V^{se}(\theta^*)$ is the unconditional variance-covariance matrix of the *simulation error*, also multiplied by instruments and the inverse return on the market, assuming the conditional distribution of $f'(t + \tau_h(t))$ given $f'(t)$ is known, and setting $N = 1$.)

PROOF (Outline)

$$\begin{aligned} \sqrt{T} \left\{ \frac{1}{T} \sum_{t=1}^T g^{(M(T), N)}(x(t), w(t); \theta^*) \right\} = & \\ \sqrt{T} \left\{ \frac{1}{T} \sum_{t=1}^T g(x(t); \theta^*) \right\} + \sqrt{T} \left\{ \frac{1}{T} \sum_{t=1}^T d'^{(N)}(x(t), w(t); \theta^*) \right\} & \\ - \sqrt{T} \left\{ \frac{1}{T} \sum_{t=1}^T d'^{(M(T), N)}(x(t), w(t); \theta^*) \right\}, & \end{aligned}$$

where $d'^{(N)}(x(t), w(t); \theta^*)$ and $d'^{(M(T), N)}(x(t), w(t); \theta^*)$ are (15×1) vectors with elements:

$$\begin{aligned} d_q'^{(N)}(x(t), w(t); \theta^*) = & \\ & \frac{I(t-1)}{I(t)} \left(c'^{(N)}\left(\frac{s_i(t)}{k_j(t)}, h(t, \tau_h(t)), w(t), \tau_j(t), \tau_h(t); \theta^*\right) \right. \\ & \left. - c'\left(\frac{s_i(t)}{k_j(t)}, h(t, \tau_h(t)), \tau_j(t), \tau_h(t); \theta^*\right) \right) y_q(t), \end{aligned}$$

$$d_q'^{(M(T), N)}(x(t), w(t); \theta^*) =$$

$$\begin{aligned} & \frac{I(t-1)}{I(t)} \left(c'^{(M(T),N)}\left(\frac{s_i(t)}{k_j(t)}, h(t, \tau_h(t)), w(t), \tau_j(t), \tau_h(t); \theta^* \right) \right. \\ & \left. - c'^{(N)}\left(\frac{s_i(t)}{k_j(t)}, h(t, \tau_h(t)), w(t), \tau_j(t), \tau_h(t); \theta^* \right) \right) y_q(t), \end{aligned}$$

$j = 1$ if $q = 1, \dots, 5$; $j = 2$ if $q = 6, \dots, 10$; $j = 3$ if $q = 11, \dots, 15$. By Lemma 1 and Chebyshev's inequality, the last term converges to 0 in probability. The first term converges to a random variable with distribution $N(0, V^{pe}(\theta^*))$, by assumptions (A1), (A4) and measurability of $g(x(t); \theta^*)$ as a function of $x(t)$. The second term converges to a random variable with distribution $N(0, \frac{1}{N} V^{se}(\theta^*))$, by the usual Central Limit arguments for Monte Carlo estimators and stationary, ergodic random variables.

Lemma 4:

$\frac{1}{T} \sum_{t=1}^T \{G^{(M(T),N)}(x(t), w(t); \theta(T)) - E[G(x(1); \theta^*)]\}$ converges to 0 in probability, provided $\theta(T)$ converges in probability to θ^* .

PROOF

$$\begin{aligned} & \frac{1}{T} \sum_{t=1}^T \{G^{(M(T),N)}(x(t), w(t); \theta(T)) - E[G(x(1); \theta^*)]\} = \\ & \frac{1}{T} \sum_{t=1}^T \{G^{(M(T),N)}(x(t), w(t); \theta(T)) - EG^{(M(T),N)}(x(1), w(1); \theta(T))\} \\ & + E\{G^{(M(T),N)}(x(1), w(1); \theta(T)) - G^{(N)}(x(1); w(1); \theta(T))\} \\ & + E\{G^{(N)}(x(1), w(1); \theta(T)) - G(x(1), w(1); \theta(T))\} \\ & + E\{G(x(1), w(1); \theta(T)) - G(x(1); \theta^*)\}, \end{aligned}$$

where:

$$G^{(N)}(x(1), w(1); \theta(T)) = \frac{d}{d\theta'} g^{(N)}(x(1), w(1); \theta(T)),$$

with rows:

$$\begin{aligned} & \frac{d}{d\theta'} g_q^{(N)}(x(1), w(1); \theta(T)) = -\frac{1}{N} \sum_{h=1}^N 1_{\{f'_n(1+\tau_j(1)) > 1\}} \cdot \frac{I(0)}{I(1)} \cdot y_q(1) \\ & \cdot \frac{d}{d\theta'} \{A(\tau_j(1)) e^{-B(\tau_j(1))r(h(1, \tau_h(1)), \tau_h(1); \theta(T))} \cdot (f'_n(1 + \tau_j(1)) - 1)\}, \end{aligned}$$

$j = 1$ if $q = 1, \dots, 5$; $j = 2$ if $q = 6, \dots, 10$; $j = 3$ if $q = 11, \dots, 15$. Uniform convergence of the second term follows from Lemma 2. The third term equals zero by the unbiasedness of the sample expectation. Continuity of $E[G(x(t); \theta)]$ at θ^* generates convergence of the last term. Consequently, Lemma 4 will hold if we can show uniform convergence in probability of the first term. We shall check the conditions of Andrews' [1987] corollary 3. $G^{(M(T), N)}(x(t), w(t); \theta(T))$ is almost surely continuous as a function of $x(t), w(t)$ and $\theta(T)$. Since $x(t)$ and $\theta(T)$ are bounded, the second condition of the corollary, essentially boundedness of $G^{(M(T), N)}(x(t), w(t); \theta(T))$, would be verified if $w(t)$ lived in a compact space. Unfortunately, it does not. Rather than bluntly assuming boundedness of $G^{(M(T), N)}(x(t), w(t); \theta(T))$, one could appeal to the following *economic* argument. Let \bar{f} be a large number such that $P\{\sup_{0 \leq u \leq \tau_j(t)} f'(t+u) \geq \bar{f}\}$ is negligible, in the sense that an option that paid $\max(\sup_{0 \leq u \leq \tau_j(t)} f'(t+u) - \bar{f}, 0)$ would have little value. Consider then the call options to have a payoff equal to:

$$\max(f'(t + \tau_j(t)) - 1, 0) - \max(\sup_{0 \leq u \leq \tau_j(t)} f'(t+u) - \bar{f}, 0).$$

The difference between the value of this option and the one with payoff $\max(f'(t + \tau_j(t)) - 1, 0)$ is economically irrelevant for large \bar{f} . Moreover, we now need only consider bounded Brownian motions. In other words, we can introduce an absorbing barrier above which no sample path moves. The boundedness of the resulting Brownian motions then leads to the desired boundedness of $G^{(M(T), N)}(x(t), w(t); \theta(T))$.

Lemma 5:

$\frac{1}{T} \sum_{t=1}^T \{\Delta^{(M(T), N)}(x(t), w(t); \theta(T), \theta^*) - E[G(x(1); \theta^*)]\}$ converges in probability to 0, provided $\theta(T)$ converges in probability to θ^* .

PROOF (Outline)

Parallels the proof of Lemma 3, with $\tilde{\lambda}_{qtn}$ substituted for $1_{\{f_n^{(M(T))}(t+\tau_j(t)) > 1\}}$.

We are now ready to demonstrate the two major results.

Theorem 1:

$\theta(T)$ converges in probability to θ^* as $T \rightarrow \infty$.

PROOF (Outline)

It is sufficient³⁰ to show uniform convergence in probability of:

³⁰See, e.g., Amemiya [1985], p. 107.

$$\frac{1}{T} \sum_{t=1}^T g^{(M(T), N)}(x(t), w(t); \theta) - E[g(x(1); \theta)].$$

For $q = 1, \dots, 15$:

$$\begin{aligned} & \left| \frac{1}{T} \sum_{t=1}^T g_q^{(M(T), N)}(x(t), w(t); \theta) - E[g_q(x(1); \theta)] \right| \leq \\ & \left| \frac{1}{T} \sum_{t=1}^T \frac{I(t-1)}{I(t)} \frac{c_i^o(t, k_j(t), \tau_j(t))}{k_j(t)} y_q(t) - E\left[\frac{I(0)}{I(1)} \frac{c_i^o(1, k_j(1), \tau_j(1))}{k_j(1)} y_q(1)\right] \right| \\ & + \left| \frac{1}{T} \sum_{t=1}^T \frac{I(t-1)}{I(t)} c'^{(M(T), N)}\left(\frac{s_i(t)}{k_j(t)}, h(t, \tau_h(t)), w(t), \tau_j(t), \tau_h(t); \theta\right) y_q(t) \right. \\ & - E\left[\frac{I(0)}{I(1)} c'^{(M(T), N)}\left(\frac{s_i(1)}{k_j(1)}, h(1, \tau_h(1)), w(1), \tau_j(1), \tau_h(1); \theta\right) y_q(1)\right] \left. \right| \\ & + E\left[\left|\frac{I(0)}{I(1)} E_{x(1)}\left[c'^{(M(T), N)}\left(\frac{s_i(1)}{k_j(1)}, h(1, \tau_h(1)), w(1), \tau_j(1), \tau_h(1); \theta\right) \right. \right. \right. \\ & - c'^{(N)}\left(\frac{s_i(1)}{k_j(1)}, h(1, \tau_h(1)), w(1), \tau_j(1), \tau_h(1); \theta\right) \left. \left. \right] \right| y_q(1) \right] \left. \right| \\ & + \left| E\left[\frac{I(0)}{I(1)} E_{x(1)}\left[c'^{(N)}\left(\frac{s_i(1)}{k_j(1)}, h(1, \tau_h(1)), w(1), \tau_j(1), \tau_h(1); \theta\right) \right. \right. \right. \right. \\ & - c'\left(\frac{s_i(1)}{k_j(1)}, h(1, \tau_h(1)), \tau_j(1), \tau_h(1); \theta\right) \left. \left. \right] y_q(1) \right] \left. \right|, \end{aligned}$$

with $j = 1$ if $q = 1, \dots, 5$; $j = 2$ if $q = 6, \dots, 10$; $j = 3$ if $q = 11, \dots, 15$. The first term on the right-hand side converges by (A1). The second term converges uniformly by the arguments of the proof of lemma 4. The third term converges by lemma 1; and the fourth term equals 0 by the unbiasedness of the sample expectation.

Theorem 2: $\sqrt{T}(\theta(T) - \theta^*)$ converges weakly to a normally distributed random vector with mean 0 and variance-covariance:

$$\begin{aligned} & (E[G(x(t); \theta^*)]' D^* E[G(x(t); \theta^*)])^{-1} E[G(x(t); \theta^*)]' D^* \Phi^{(N)}(\theta^*) \\ & D^* E[G(x(t); \theta^*)] (E[G(x(t); \theta^*)]' D^* E[G(x(t); \theta^*)])^{-1}. \end{aligned}$$

PROOF (Outline).

Multiply the expansion in (23) by $[\frac{1}{T} \sum_{t=1}^T G^{(M(T),N)}(x(t), w(t); \theta(T))] D(T)$. The left-hand side equals zero by the first-order conditions that define the MSM estimator. A rearrangement produces:

$$\begin{aligned}
& [\frac{1}{T} \sum_{t=1}^T G^{(M(T),N)}(x(t), w(t); \theta(T))] D(T) \\
& \quad [\frac{1}{T} \sum_{t=1}^T \Delta^{(M(T),N)}(x(t), w(t); \theta(T), \theta^*)] \sqrt{T}(\theta(T) - \theta^*) \\
& = -[\frac{1}{T} \sum_{t=1}^T G^{(M(T),N)}(x(t), w(t); \theta(T))] D(T) \\
& \quad \cdot \sqrt{T}(\frac{1}{T} \sum_{t=1}^T g^{(M(T),N)}(x(t), w(t); \theta^*)). \tag{24}
\end{aligned}$$

The right-hand side equals:

$$\begin{aligned}
& -\{\frac{1}{T} \sum_{t=1}^T G^{(M(T),N)}(x(t), w(t); \theta(T))' - EG(x(t); \theta^*)'\} D(T) \\
& \quad + EG(x(t); \theta^*)'(D(T) - D^*) + EG(x(t); \theta^*)' D^* \\
& \quad \{\sqrt{T}(\frac{1}{T} \sum_{t=1}^T g^{(M(T),N)}(x(t), w(t); \theta^*)\}.
\end{aligned}$$

The first term in the first factor converges to 0 in probability, by lemma 4. The second term converges by assumption. The second factor converges to a normally distributed random vector with mean 0 and variance-covariance $\Phi^{(N)}(\theta^*)$, by Lemma 3. The right-hand side in (24) can be rearranged to give:

$$\begin{aligned}
& \{[\frac{1}{T} \sum_{t=1}^T (G^{(M(T),N)}(x(t), w(t); \theta(T))' - EG(x(t); \theta^*)') D(T) \\
& \quad + EG(x(t); \theta^*)'(D(T) - D^*)] \frac{1}{T} \sum_{t=1}^T \Delta^{(M(T),N)}(x(t), w(t); \theta(T), \theta^*) \\
& \quad + EG(x(t); \theta^*)' D^* \frac{1}{T} \sum_{t=1}^T [\Delta^{(M(T),N)}(x(t), w(t); \theta(T), \theta^*) - EG(x(t); \theta^*)] \\
& \quad + EG(x(t); \theta^*)' D^* EG(x(t); \theta^*)\} \sqrt{T}(\theta(T) - \theta^*).
\end{aligned}$$

The first three terms of the first factor converge to 0 in probability by Lemma 4, by

assumption, and by Lemma 5, respectively. The last term can be inverted to generate the desired result.

Table 1a

Descriptive Statistics

<i>variable</i>	<i>Bank of America</i>		
	<i>mean</i>	<i>min</i>	<i>max</i>
$s_i(t)$	15.809	12.000	18.380
$k_1(t)$	16.463	10.000	20.000
$k_2(t)$	16.458	10.000	20.000
$k_3(t)$	16.559	10.000	20.000
$\tau_1(t)$	0.2366	0.0028	0.6694
$\tau_2(t)$	0.2370	0.0028	0.6694
$\tau_3(t)$	0.2395	0.0028	0.6694
$c_i^o(t, k_1(t), \tau_1(t))$	1.237	0.060	7.750
$c_i^o(t, k_2(t), \tau_2(t))$	1.244	0.060	8.000
$c_i^o(t, k_3(t), \tau_3(t))$	1.224	0.060	13.000
$I(t)/I(t-1)$	0.99997	0.99162	1.00654
$\tau_h(t)$	0.2903	0	1.9778
$h(t, \tau_h(t))$	0.9840	0.9792	0.9870
$\text{lag}(I)$	3.5	0	51.0
$\text{lag}(h)$	193.7	0	893.0
$\text{lag}(c_{i1}^o)$	159.3	0	845.0
$\text{lag}(h, I)$	190.2	-30.0	892.0
$\text{lag}(c_{i1}^o, I)$	155.8	-24.0	845.0
$\text{lag}(c_{i1}^o, h)$	-34.4	-884.0	813.0
$\text{lag}(c_{i3}^o, c_{i1}^o)$	194.8	7.0	840.0
$s_{i3} - s_{i1}$	-0.002	-0.250	0.500

Remarks about Table 1a are below Table 1b.

Table 1b

Descriptive Statistics

variable	Federal Express			Bethlehem Steel		
	mean	min	max	mean	min	max
$s_i(t)$	63.398	55.250	73.130	17.944	14.130	22.000
$k_1(t)$	64.206	35.000	75.000	18.568	12.500	22.500
$k_2(t)$	64.123	35.000	75.000	18.617	10.000	22.500
$\tau_1(t)$	0.1619	0.0028	0.6500	0.2912	0.0028	0.7528
$\tau_2(t)$	0.1579	0.0028	0.6500	0.2950	0.0028	0.7444
$c_i^o(t, k_1(t), \tau_1(t))$	3.740	0.060	27.000	1.468	0.060	7.500
$c_i^o(t, k_2(t), \tau_2(t))$	3.730	0.060	26.750	1.473	0.060	7.500
$I(t)/I(t-1)$	0.99996	0.99162	1.00654	1.00004	0.99278	1.00505
$\tau_h(t)$	0.2959	0	1.9778	0.2977	0	1.6889
$h(t, \tau_h(t))$	0.9842	0.9792	0.9870	0.9842	0.9794	0.9870
$\text{lag}(I)$	4.0	0	378.0	3.3	0	51.0
$\text{lag}(h)$	195.4	0	892.0	193.2	0	893.0
$\text{lag}(c_{i1}^o)$	206.6	0	879.0	284.5	0	891.0
$\text{lag}(h, I)$	191.4	-227.0	892.0	189.9	-30.0	892.0
$\text{lag}(c_{i1}^o, I)$	202.6	-38.0	869.0	281.2	-16.0	891.0
$\text{lag}(c_{i1}^o, h)$	11.2	-831.0	869.0	91.2	-856.0	870.0
$\text{lag}(c_{i2}^o, c_{i1}^o)$	136.4	0	870.0	137.1	3.0	791.0
$s_{i2} - s_{i1}$	0.005	-0.750	0.870	-0.003	-0.500	0.250

Remarks about Table 1. The table displays summary statistics of a dataset consisting of Chicago Board Options Exchange stock call option transactions and Chicago Mercantile Exchange 90-day Treasury bill futures transactions at 15-minute intervals between 1 January 1986 and 30 June 1986. The argument t refers to the t th observation. Total number of observations = 1629 (*Bank of America*), 1556 (*Federal Express*), 665 (*Bethlehem Steel*). $s_i(t)$ = stock price observed simultaneous with the first call option transaction after the t th 15-minute mark; $k_1(t)$, $k_2(t)$, $k_3(t)$ = exercise price of the first, second and third call option transaction after the t th 15-minute mark, respectively; $\tau_1(t)$, $\tau_2(t)$, $\tau_3(t)$ = time-to-maturity (as a fraction of one year) of the first, second and third call option transaction after the t th 15-minute mark, respectively; $c_i^o(t, k_1(t), \tau_1(t))$, $c_i^o(t, k_2(t), \tau_2(t))$, $c_i^o(t, k_3(t), \tau_3(t))$ = price of the first, second and third call option transaction after the t th 15-minute mark, respectively; $I(t)/I(t-1)$ = return on the S&P100 index over the 15-minute period preceeding the t th 15-minute mark; $\tau_h(t)$ = time-to-maturity (as a fraction of one year) of the first 90-day Treasury bill futures transaction after the t th 15-minute mark; $h(t, \tau_h(t))$ = quote of the first 90-day Treasury bill futures transaction after the t th 15-minute mark; $\text{lag}(I)$ = time (in seconds) between 15-minute mark and first observation of a S&P100 quote; $\text{lag}(h)$ = time (in seconds) between 15-minute mark and first Treasury bill futures transaction; $\text{lag}(c_{i1}^o)$ = time (in seconds) between 15-minute mark and first call option transaction; $\text{lag}(h, I)$ = time (in seconds) between

first Treasury bill futures transaction and first observation of an S&P100 quote; $\text{lag}(c_{i,1}^o, l)$ = time (in seconds) between first call option transaction and first observation of an S&P100 quote; $\text{lag}(c_{i,1}^o, h)$ = time (in seconds) between first call option transaction and first Treasury bill futures transaction; $\text{lag}(c_{i,3}^o, c_{i,1}^o)$ = time (in seconds) between third call option transaction and first call option transaction; $\text{lag}(c_{i,2}^o, c_{i,1}^o)$ = time (in seconds) between second call option transaction and first call option transaction; $s_{i,3} - s_{i,1}$ = difference in the stock price observed simultaneous with the third call option transaction and the one observed simultaneous with the first call option transaction; $s_{i,2} - s_{i,1}$ = difference in the stock price observed simultaneous with the second call option transaction and the one observed simultaneous with the first call option transaction. All prices are in US dollar. Data sources: *Berkeley Options Data Base* (call option transactions data, S&P100 data), *Chicago Mercantile Exchange* (futures transactions data).

Table 2a

Discretization-Induced Biases
 $\rho_i = 0.00$

M	<i>exercise price</i>			
	<i>10.000</i>	<i>13.000</i>	<i>16.463</i>	<i>20.000</i>
2	5.939	2.981	0.266	<0.001
4	5.942	2.984	0.264	<0.001
8	5.953	2.995	0.274	0.001
16	5.940	2.982	0.266	0.001
32	5.936	2.978	0.260	0.001
64	5.937	2.978	0.264	<0.001
128	5.956	3.002	0.276	0.001
256	5.924	2.967	0.259	0.001

Table 2b

Discretization-Induced Biases
 $\rho_i = 0.50$

M	<i>exercise price</i>			
	<i>10.000</i>	<i>13.000</i>	<i>16.463</i>	<i>20.000</i>
2	5.948	2.989	0.291	0.003
4	5.967	3.008	0.303	0.003
8	5.964	3.006	0.302	0.003
16	5.943	2.985	0.286	0.002
32	5.945	2.987	0.284	0.003
64	5.951	2.993	0.293	0.003
128	5.935	2.976	0.271	0.002
256	5.960	3.002	0.298	0.003

Remarks about Tables 2a and 2b. The tables display estimated Cox-Ingersoll-Ross call prices when the stock price equals 15.809 (the average stock price for *Bank of America*; see Table 1a), the time-to-maturity equals 0.2366 (the average time-to-maturity for *Bank of America*), and the exercise price is as indicated on top of each column (16.463 is the average exercise price for *Bank of America*). In Table 2a, the correlation between the stock price and the interest rate (ρ_i) is zero. In Table 2b, it equals 0.50. Other parameter values: $\sigma'_i = .6$; $\sigma_x = .25$; $\alpha = .5$; $\beta' = .1$. The continuous-time interest rate is backed out of the average 90-day Treasury bill futures quote for *Bank of America* (0.9840), using the average time-to-maturity (0.2903). M is the number of intervals over which the path of the stock price and the continuous-time interest rate are discretized. Romberg interpolation, as described in Talay and Tubaro [1989],

was used to minimize discretization biases. In each estimation, 10000 paths were simulated (i.e., $N = 10000$). The same random numbers were used across columns, but not across rows. For comparison, the Black-Scholes call prices evaluated at the interest rate implicit in the Treasury bill futures price (.056) equal 5.941, 2.981, 0.259 and less than 0.001, for exercise prices equal to 10.000, 13.000, 16.463 and 20.000, respectively.

Table 3

The Size of the Simulation Error

	<i>standard error</i>			
	<i>pricing</i>	<i>simulation</i> (<i>N=1</i>)	<i>check</i>	
			<i>a</i>	<i>b</i>
<i>Bank of America:</i>				
<i>1st series</i>	.02156	.01345	.02160	.02160
<i>2nd series</i>	.02189	.01142	.02192	.02188
<i>3rd series</i>	.03207	.01407	.03210	.03205
<i>Federal Express:</i>				
<i>1st series</i>	.01668	.00867	.01670	.01674
<i>2nd series</i>	.01477	.00986	.01480	.01490
<i>Bettehem Steel:</i>				
<i>1st series</i>	.02256	.01567	.02261	.02234
<i>2nd series</i>	.02299	.01595	.02305	.02322

Remarks about Table 3. For each series in the dataset, this table displays standard errors calculated from the difference between the observed call price and the estimated Cox-Ingersoll-Ross call price normalized by the exercise price. The *pricing error* was obtained directly from estimating the Cox-Ingersoll-Ross call price using 200 simulations per observation (i.e., $N=200$), thereby assuming that the simulation error generated with 200 simulations is negligible. The *simulation error* for $N=1$ was obtained from subtracting the total error obtained using 200 simulations from the one obtained using 1 simulation (i.e., $N=1$). The assumption that the simulation error with 200 simulations is negligible was *checked* by comparing the standard deviation of the total error for 100 simulations computed from the data (*a*) to the one obtained by taking the square root of the sum of the first column (squared) and 1/100 times the second column (squared) (*b*). If the assumption is correct, the standard errors for $N=100$ computed either way should be of similar magnitude. Different random draws were taken in each of the estimations. The simulation error was minimized by correlating the difference between the observed and the estimated call price with the simulation error obtained from the Black-Scholes model (variance reduction). M , the number of intervals over which the path of the stock price and the continuous-time interest rate are discretized, was set equal to 8. Romberg interpolation, as described in Talay and Tubaro [1989], was used to minimize discretization biases. Parameter values: $\sigma'_t = .6$; $\sigma_x = .25$; $\rho_t = .0$; $\alpha = .5$; $\beta' = .1$.

Table 4

Method of Simulated Moments Estimation Results

	<i>restricted model</i> ($\sigma_z = 0, \alpha = 0$)		
	<i>Bank of America</i>	<i>Federal Express</i>	<i>Bellehem Steel</i>
$\hat{\sigma}_i'$	1.650	1.454	1.270
$stderr(\hat{\sigma}_i')$	0.006	0.001	0.004
χ^2	329.6	242.0	305.0
<i>prob value</i>	<.001	<.001	<.001
<i>dof</i>	14	9	9
<i>mean pricing error:</i>			
<i>1st series</i>	-0.020	-0.004	0.007
<i>2nd series</i>	-0.015	0.002	0.015
<i>3rd series</i>	0.002		
<i>standard deviation:</i>			
<i>1st series</i>	0.172	0.253	0.567
<i>2nd series</i>	0.173	0.251	0.511
<i>3rd series</i>	0.339		
	<i>restricted model</i> ($\sigma_z = 0.001, \rho_i = 0, \alpha = 1$)		
	<i>Bank of America</i>	<i>Federal Express</i>	<i>Bellehem Steel</i>
$\hat{\sigma}_i'$	1.512	1.462	1.466
$stderr(\hat{\sigma}_i')$	<0.001	0.009	0.037
$\hat{\beta}'$	0.035	0.100	0.066
$stderr(\hat{\beta}')$	0.005	<0.001	0.007
$corr(\hat{\sigma}_i', \hat{\beta}')$.954	.136	.939
χ^2	295.7	257.3	250.3
<i>prob value</i>	<.001	<.001	<.001
<i>dof</i>	13	8	8
<i>mean pricing error:</i>			
<i>1st series</i>	-0.002	0.001	-0.005
<i>2nd series</i>	-0.001	0.001	-0.083
<i>3rd series</i>	0.005		
<i>standard deviation:</i>			
<i>1st series</i>	0.233	0.753	0.267
<i>2nd series</i>	0.227	1.027	0.266
<i>3rd series</i>	0.362		

Table 4 (continued)

Method of Simulated Moments Estimation Results

	<i>Black-Scholes model (1)</i> ($\hat{\sigma}_i' = 0, \hat{R} = 0$)		
	<i>Bank of America</i>	<i>Federal Express</i>	<i>Bethlehem Steel</i>
χ^2	1365.9	533.4	1224.00
prob value	<.001	<.001	<.001
dof	15	10	10
mean pricing error:			
1st series	0.700	1.905	0.917
2nd series	0.713	1.892	0.938
3rd series	0.725		
standard deviation:			
1st series	0.505	1.622	0.641
2nd series	0.518	1.537	0.662
3rd series	0.588		
	<i>Black-Scholes model (2)</i>		
	<i>Bank of America</i>	<i>Federal Express</i>	<i>Bethlehem Steel</i>
χ^2	95.6	26.3	11.4
prob value	<.001	.003	.329
dof	15	10	10
mean pricing error:			
1st series	0.011	-0.033	-0.004
2nd series	0.008	-0.025	-0.003
3rd series	0.018		
standard deviation:			
1st series	0.156	0.576	0.200
2nd series	0.155	0.501	0.205
3rd series	0.333		

Remarks about Table 4. This table displays results from Method of Simulated Moments estimation of various call option pricing models on the three datasets described in Table 1. The continuous-time processes were approximated over 8 intervals (i.e., $M = 8$), and 10 simulations were performed per observation in the dataset (i.e., $N = 10$) (40 simulations per observation were used when calculating the weighting matrix for the second stage of Method of Simulated Moments estimation). Variance reduction and Romberg interpolation were employed as in Table 3. Generalized Method of Moments was used whenever an analytical pricing formula existed. The following symbols are used in the table: $\hat{\sigma}_i'$ = estimated volatility parameter of the stock price process; $\hat{\beta}'$ = estimated long-run interest rate; \hat{R} = estimated

continuous-time interest rate; *stderr*(.) = asymptotic standard error of a parameter estimate; *corr*(., .) = asymptotic correlation between two parameter estimates; $\chi^2 = \chi^2$ -statistic that tests the over-identifying restrictions; *prob value* = probability level of the χ^2 -statistic under the null hypothesis; *dof* = number of degrees of freedom in the χ^2 -test (equal to the number of moment conditions minus the number of parameters to be estimated). The *pricing error* is *not* normalized by the exercise price; it is expressed in US dollar. In the *Black-Scholes model (1)*, the stock price volatility and the interest rate were constrained to be constant over time. In the *Black-Scholes model (2)*, the stock price volatility is implied from the previous pricing error in the dataset, and the interest rate is implied from the Treasury bill futures price assuming a flat term structure.

Table 5

Regression of the Pricing Errors onto the Instruments

	pricing error	intercept	instruments			
			1st	2nd	3rd	4th
Bank of America:						
1st series						
mean	-0.0016		0.9736	0.2356	0.98403	0.0826
standard deviation	0.0109		0.1307	0.1677	0.0014	0.0837
coefficient estimate		-3.1069	-0.0072	-0.0013	3.1651	0.0110
standard error		0.1749	0.0019	0.0015	0.1778	0.0029
2nd series						
mean	-0.0013		0.9735	0.2370	0.9840	0.0827
standard deviation	0.0110		0.1279	0.1670	0.0014	0.0810
coefficient estimate		-3.1950	-0.0075	-0.0092	3.2549	0.0031
standard error		0.1797	0.0020	0.0015	0.1826	0.0030
3rd series						
mean	-0.0001		0.9680	0.2395	0.9840	0.0815
standard deviation	0.0259		0.1292	0.1641	0.0014	0.0859
coefficient estimate		-2.3125	-0.0027	-0.0108	2.3550	0.0033
standard error		0.4636	0.0050	0.0040	0.4711	0.0074
Federal Express:						
1st series						
mean	0.0000		0.9959	0.1618	0.9842	0.0635
standard deviation	0.0104		0.1048	0.1344	0.0013	0.0765
coefficient estimate		-1.9754	-0.0054	-0.0031	2.0127	0.0058
standard error		0.2013	0.0026	0.0019	0.2037	0.0034
2nd series						
mean	0.0002		0.9973	0.1578	0.9842	0.0637
standard deviation	0.0085		0.1059	0.1354	0.0013	0.0788
coefficient estimate		-1.7176	-0.0012	-0.0070	1.7476	0.0023
standard error		0.1592	0.0020	0.0016	0.1612	0.0027
Bethlehem Steel:						
1st series						
mean	-0.0001		0.9756	0.2916	0.9842	0.0843
standard deviation	0.0144		0.1237	0.1885	0.0013	0.0786
coefficient estimate		-5.565	-0.0021	0.0027	5.6560	-0.0034
standard error		0.3734	0.0041	0.0026	0.3784	0.0062
2nd series						
mean	0.0002		0.9736	0.2946	0.9842	0.0845
standard deviation	0.0145		0.1240	0.1872	0.0013	0.0778
coefficient estimate		-5.6525	-0.0032	0.0033	5.7461	-0.0048
standard error		0.3742	0.0040	0.0026	0.3792	0.0063

Remarks about Table 5. Estimation results are displayed from Ordinary Least Squares regression of the pricing error (normalized by the exercise price) divided by the market return onto the instruments. The pricing errors were obtained at the Generalized Method of Moments estimates for the model with $\sigma_x = 0$ and $\rho_1 = 0$ (see first panel of Table 4). Means and standard deviations of the pricing error and the instruments are also shown. The instruments are: stock price lagged one period divided by current exercise price (1st); time-to-maturity (2nd); Treasury bill futures quote lagged one period (3rd); call price divided by exercise price, lagged one period (4th). The F -statistics for the regressions, in the order as they appear in the table, are: 116.0, 100.5, 9.1, 32.2, 39.1, 61.3, 64.9. The corresponding probability levels are all below .001.

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